ON AN EXACT FORMULA FOR THE COEFFICIENTS OF HAN'S GENERATING FUNCTION

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Abstract. Given a positive integer t and a partition λ, define $\mathcal{H}_t(\lambda)$ to be the multiset of hook lengths of λ that are divisble by t. For each nonnegative integer n, we consider the quantity $a_t(n) = a_t^{even}(n) - a_t^{odd}(n)$, where $a_t^{even}(n)$ (resp. $a_t^{odd}(n)$) is the number of partitions λ of n for which $\mathcal{H}_t(\lambda)$ has an even (resp. odd) number of elements. We prove an exact formula for $a_t^{even}(n) - a_t^{odd}(n)$ using a generating function for $a_t(n)$ discovered by Han in his generalization of the Nekrasov-Okounkov formula. Moreover, we obtain corollaries which describe the asymptotic behavior and sign of $a_t(n)$ for large n.

1. Introduction

One of the most important functions in number theory and combinatorics is the partition function $p(n)$, which is known to have the following generating function:

$$f(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{j=0}^{\infty} p(j)x^j.$$

One celebrated achievement in analytic number theory was the determination of an asymptotic relation for $p(n)$. In 1918, Hardy and Ramanujan [5] discovered that

$$p(n) \sim \frac{e^{\pi(2/3)^{1/2}\sqrt{n}}}{4n\sqrt{3}}. \tag{1.1}$$

In doing so, they developed what came to be known as the Hardy-Littlewood *circle method*, which has become an important tool for analytic number theorists in solving problems dealing with asymptotics. The method was famously perfected by Rademacher [9, 10], who improved the result of Hardy and Ramanujan. He was able to obtain an exact formula for $p(n)$:

$$p(n) = \sum_{k=1}^{\infty} \rho_k(n), \tag{1.2}$$

with

$$\rho_k(n) = \frac{(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} A_k(n) k^{-1} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right),$$

where I_v is a modified Bessel function of the first kind and

$$A_k(n) = \sum_{\substack{0\le h<k \\ (h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k},$$

where $\omega_{h,k}$ is given by (2.4). One can observe that $\rho_1(n)$ dominates the sum (1.2), and using the approximation

$$I_v(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{1.3}$$

for large x (p. 204 of [6]), one can retrieve (1.1) from (1.2).

What is remarkable about (1.2) is how rapidly the sum converges to $p(n)$. Let $p_k(n)$ be the k^{th} partial sum of (1.2), namely,

$$p_k(n) := \sum_{j=1}^{k} \rho_j(n).$$

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A table showing the convergence for various values of n is displayed below.

	$n = 10$	$n = 25$	$n = 50$	$n = 100$
$p_1(n)$	41.6278	1960.0983	204211.0765	190568944.7833
$p_2(n)$	42.0697	1958.0059	204226.7970	190569293.6549
$p_3(n)$	42.0157	1957.8805	204226.0430	190569291.0568
$p_4(n)$	41.9618	1957.9181	204225.8495	190569291.7423
$p_5(n)$	42.0146	1957.9932	204225.9768	190569292.0599
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p(n)$	42	1958	204226	190569292

TABLE 1. Rapid convergence of Rademacher's formula for the number of partitions

Now, it is known that the number of partitions of an integer n is the same as the number of irreducible representations of the symmetric group S_n [11]. Moreover, in representation theory, one is often concerned with the study of relations involving hook lengths of partitions [7, 11]. Recall that a hook length of a box in the Young diagram of a partition is the number of boxes to the right of or below the box plus the box itself. For example, Figure 1 displays the hook lengths for the partition $(6, 4, 3, 1)$.

9	7	6	4	2	1
6	4	3	1		
4	2	1			
1					

FIGURE 1. Hook lengths for the partition $(6, 4, 3, 1)$

In this paper, we will concern ourselves with hook lengths which are a multiple of a certain number. In particular, for any integer $t \geq 1$ and a partition λ, define $\mathcal{H}_t(\lambda)$ to be the multiset of hook lengths of λ which are divisible by t. Moreover, consider the following power series:

$$F_t(x) := \sum_{n=0}^{\infty} a_t(n) x^n := \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} (-1)^{\#\mathcal{H}_t(\lambda)}, \tag{1.4}$$

where $\mathcal{P}$ is the set of all partitions (including the null partition). Observe that $a_t(n) = a_t^{even}(n) - a_t^{odd}(n)$, where $a_t^{even}(n)$ (resp. $a_t^{odd}(n)$) is the number of partitions of n having an even (resp. odd) number of hook lengths divisible by t.

Remark. It is interesting to note that the special case $t = 1$ results in

$$F_1(x) = \sum_{n \geq 0} (-1)^n p(n) x^n,$$

meaning that $a_1(n) = (-1)^n p(n)$.

One can list the numbers $a_t(n)$ for fixed t in hope of finding patterns. Consider, for instance, the case $t = 2$. A list of the numbers $a_2(n)$ for the first few n is shown in Table 2. Upon inspection of Table 2, one notices that $|a_t(n)|$ seems to be weakly increasing with n. Moreover, the $a_t(n)$ appear to follow a sign pattern, namely, the signs repeat with periodicity 4.

In light of these observations, it is natural to ask for asymptotics and sign patterns of $a_t(n)$ for general t. In this paper, we give an exact formula for $a_t(n)$ which is stated in Theorem 1.1.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$a_2(n)$	1	1	-2	-1	5	3	-9	-5	18	10	-30	-16	53	29	-85	-44	139	73	-215	-110	335

TABLE 2. The numbers $a_2(n)$ for various n

Let us introduce some notation. For any positive integer u, define $\mathrm{ord}_2(u)$ to be the unique nonnegative integer m such that $2^m|u$ but $2^{m+1}\nmid u$. Then, we have the following theorem:

Theorem 1.1. *If $t \geq 1$ and $n \geq 1$, then*

$$a_t(n) = \sum_{\substack{1\leq k\\(k,2t)=1}} \sum_{\substack{0\leq h<k\\(h,k)=1}} \sum_{m=0}^{M(t,k)} g_t^{(1)}(h,k,m,n) + \sum_{\substack{1\leq k\\c=d-1}} \sum_{\substack{0\leq h<k\\(h,k)=1}} \sum_{m=0}^{M(t,k)} g_t^{(2)}(h,k,m,n) + \sum_{\substack{1\leq k\\c\leq d-2}} \sum_{\substack{0\leq h<k\\(h,k)=1}} \sum_{m=0}^{M(t,k)} g_t^{(3)}(h,k,m,n), \tag{1.5}$$

where $g_t^{(1)}$, $g_t^{(2)}$, and $g_t^{(3)}$ are given by

$$g_t^{(1)}(h,k,m,n) = k^{-1}e^{2\pi i(mh'/tk-nh/k)}\Omega_{t,h,k}b_{t,h,k}\frac{2^{t/2}(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} \times \left(\frac{1}{4}-\frac{24m}{t}\right)^{3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(\frac{1}{4}-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)}\right), \tag{1.6}$$

$$g_t^{(2)}(h,k,m,n) = k^{-1}e^{2\pi i(mh'/tk-nh/k)}\Omega_{t,h,k}b_{t,h,k}(m)\frac{2^{-t/2}(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} \times \left(3\cdot 2^{2c}r^2+1-\frac{24m}{t}\right)^{3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(3\cdot 2^{2c}r^2+1-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)}\right), \tag{1.7}$$

and

$$g_t^{(3)}(h,k,m,n) = k^{-1}e^{2\pi i(mh'/tk-nh/k)}\Omega_{t,h,k}b_{t,h,k}(m)\frac{(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} \times \left(1-\frac{24m}{t}\right)^{3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(1-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)}\right), \tag{1.8}$$

where $c := \mathrm{ord}_2(t)$, $d := \mathrm{ord}_2(k)$, $r := \left(\frac{t}{2^c}, \frac{k}{2^d}\right)$, and $hh' \equiv -1 \pmod{k}$. Moreover, $\Omega_{t,h,k}$ is defined by (2.7) and $b_{t,h,k}(m)$ is defined by (3.8), while $M(t,k)$ is defined by (3.10) for $c \geq d$, (3.24) for $c = d-1$, and (3.31) for $c \leq d-2$.

Since $I_{3/2}(x)$ is given by

$$I_{3/2}(x) = \sqrt{\frac{2x}{\pi}}\frac{d}{dx}\left(\frac{\sinh x}{x}\right), \tag{1.9}$$

we obtain the following corollary concerning asympototics for $a_t(n)$.

Corollary 1.2. *Given a positive integer t, for positive n we have*

$$a_t(n) = \frac{3^{-1/2}(3\cdot 2^{2c}+1)^{1/2}B_{2^{c+1}}(n)}{2^{\frac{c}{2}+\frac{t}{2}+\frac{5}{2}}\left(n-\frac{1}{24}\right)} e^{\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}}\left(1+o(1)\right)$$

as $n\to\infty$, where c is the nonnegative integer for which $c = \mathrm{ord}_2(t)$ and

$$B_{2^{c+1}}(n) = \sum_{\substack{0\leq h<2^{c+1}\\ h \text{ odd}}} \exp\left\{i\pi\left(3ts\left(\frac{th}{2^c},1\right)+s(h,2^{c+1})-2ts\left(\frac{th}{2^c},2\right)-ts\left(\frac{th}{2^{c-1}},1\right)\right)\right\}e^{-\frac{i\pi nh}{2^c}}, \tag{1.10}$$

where s is the usual Dedekind sum, as defined in (2.3).

Now, we define a positive integer t to be *good* if the corresponding Kloosterman sum (1.10) is nonzero for all n. Then, we have the following corollary:

Corollary 1.3. *Suppose t is a good positive integer. Then, for n sufficiently large, if $a_t(n)$ is positive (resp. negative), then $a_t(n+2^c)$ is negative (resp. positive). In particular, the signs of $a_t(n)$ are eventually periodic modulo 2^{c+1}.*

Remark. It is believed that all positive integers are good. All integers $1 \le t \le 1000$ have been verified to be good (note that (1.10) need only be verfied to be nonzero for $0 \le n < 2^{c+1}$, since $B_{2^{c+1}}$ is periodic with period 2^{c+1}).

The formula (1.5) exhibits rather rapid convergence. For instance, let

$$s_{t,N}(n) = \sum_{\substack{1\le k\le N \\ (k,2t)=1}} \sum_{\substack{0\le h<k \\ (h,k)=1}} \sum_{m=0}^{M(t,k)} g_t^{(1)}(h,k,m,n) + \sum_{\substack{1\le k\le N \\ c=d-1}} \sum_{\substack{0\le h<k \\ (h,k)=1}} \sum_{m=0}^{M(t,k)} g_t^{(2)}(h,k,m,n) + \sum_{\substack{1\le k\le N \\ c\le d-2}} \sum_{\substack{0\le h<k \\ (h,k)=1}} \sum_{m=0}^{M(t,k)} g_t^{(3)}(h,k,m,n) \tag{1.11}$$

We can consider how quickly the sequence $s_{t,1}(n), s_{t,2}(n), s_{t,3}(n), \ldots$ converges to $a_t(n)$ for various t and n. Table 3 shows the convergence for the case $t = 6$.

	$n=10$	$n=25$	$n=50$	$n=100$
$s_{6,10}(n)$	-5.2076	106.4856	-14047.1123	6301352.6325
$s_{6,20}(n)$	-18.1519	-154.0285	-1634.5136	9398574.9680
$s_{6,30}(n)$	-18.1465	-154.0252	-1634.5238	9398574.9711
$s_{6,40}(n)$	-17.9606	-154.0161	-1634.0174	9398571.8233
$s_{6,50}(n)$	-17.9606	-154.0163	-1634.0170	9398571.8227
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_6(n)$	-18	-154	-1634	9398572

TABLE 3. Rapid convergence of (1.5) for the case $t = 6$

In Section 2, we will recall a generating function of Han and use it to derive a transformation law for F_t. Then in Section 3, we will use this transformation law and invoke the circle method to prove Theorem 1.1 and Corollaries 1.2 and 1.3.

2. Properties of $F_t(x)$

2.1. **Formula for $F_t(x)$.** One of the most famous hook length formulas is the Nekrasov-Okounkov formula, which relates powers of the Euler Product to partition hook lengths [8]. This formula was generalized by Han [4], who proved the generalization using Macdonald identites and a result of Garvan, Kim, and Stanton [3] which establishes properties of a bijection between N-codings and t-cores. Han presents the following identity as a corollary to his main result (Corollary 5.2 in [4]):

$$\sum_{\lambda\in\mathcal{P}} x^{|\lambda|}(-1)^{\#\mathcal{H}_t(\lambda)} = \sum_{n=0}^{\infty} a_t(n)x^n = \prod_{k\ge1} \frac{(1-x^{4tk})^t(1-x^{tk})^{2t}}{(1-x^{2tk})^{3t}(1-x^k)}. \tag{2.1}$$

2.2. **Transformation Formula.** Define

$$f(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j}, \tag{2.2}$$

the generating function for unrestricted partitions. In terms of the Dedekind eta-function, $f(e^{2\pi i\tau}) = e^{\pi i\tau/12}/\eta(\tau)$. We make use of the following familiar notation:

Definition 2.1. *For positive integers h, k, define the Dedekind sum $s(h,k)$ by*

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right) \tag{2.3}$$

and

$$\omega_{h,k} = e^{i\pi s(h,k)}. \tag{2.4}$$

Using Dedekind's functional equation and the well-known transformational law of the Dedekind eta function, one can derive the following transformational formula for f (see [1]):

$$f\left(\exp\left(2\pi i \frac{iz+h}{k}\right)\right) = \omega_{h,k}\sqrt{z}\exp\left\{\frac{\pi}{12k}\left(\frac{1}{z} - z\right)\right\} f\left(\exp\left(2\pi i \frac{i/z + h'}{k}\right)\right), \tag{2.5}$$

in which h, h', and k are positive integers satisfying $(h,k) = 1$ and $hh' \equiv -1 \pmod{k}$. We use this transformation formula for f to obtain a transformation formula for F_t in Theorem 2.3. First, let us introduce some notation to simplify the statement of Theorem 2.3.

Definition 2.2. *For any positive integers t and k, define*

$$\psi_{t,k}(z) = \begin{cases} 2^{t/2}\exp\left\{\frac{\pi}{12k}\left(\frac{-3\cdot 2^{2d-2}r^2+1}{z} - z\right)\right\}, & c \geq d \\ 2^{-t/2}\exp\left\{\frac{\pi}{12k}\left(\frac{3\cdot 2^{2c}r^2+1}{z} - z\right)\right\}, & c = d-1 \\ \exp\left\{\frac{\pi}{12k}\left(\frac{1}{z} - z\right)\right\}, & c \leq d-2 \end{cases}, \tag{2.6}$$

where $c = \operatorname{ord}_2(t)$, $d = \operatorname{ord}_2(k)$, and $r = (t/2^c, k/2^d)$. Also, if h is a positive integer relatively prime to k, then we define

$$\Omega_{t,h,k} = \begin{cases} \dfrac{\omega^{3t}_{\frac{th}{2^{d-1}r}, \frac{k}{2^d r}}\,\omega_{h,k}}{\omega^{2t}_{\frac{th}{2^d r}, \frac{k}{2^d r}}\,\omega^{t}_{\frac{th}{2^{d-2}r}, \frac{k}{2^d r}}}, & c \geq d \\ \dfrac{\omega^{3t}_{\frac{th}{2^c r}, \frac{k}{2^{c+1} r}}\,\omega_{h,k}}{\omega^{2t}_{\frac{th}{2^c r}, \frac{k}{2^c r}}\,\omega^{t}_{\frac{th}{2^{c-1}r}, \frac{k}{2^{c+1} r}}}, & c = d-1 \\ \dfrac{\omega^{3t}_{\frac{th}{2^c r}, \frac{k}{2^{c+1} r}}\,\omega_{h,k}}{\omega^{2t}_{\frac{th}{2^c r}, \frac{k}{2^c r}}\,\omega^{t}_{\frac{th}{2^c r}, \frac{k}{2^{c+2} r}}}, & c \leq d-2 \end{cases}. \tag{2.7}$$

Now, we state the transformation formula for F_t.

Theorem 2.3. *Suppose h, k are relatively prime integers and fix $t \geq 1$. Let $c = \operatorname{ord}_2(t)$, $d = \operatorname{ord}_2(k)$, and $r = (t/2^c, k/2^d)$. Also, let h' be an integer such that $hh' \equiv -1 \pmod{k}$. Then, we have*

$$F_t\left(\exp\left(2\pi i \frac{iz+h}{k}\right)\right) = \Omega_{t,h,k}\sqrt{z}\,\psi_{t,k}(z)\,G_{t,h,k}\left(\exp\left(2\pi i \frac{i/z+h'}{tk}\right)\right), \tag{2.8}$$

where $G_{t,h,k}(x)$ is some power series in x with radius of convergence at least 1 and having constant term 1.

Proof. Note that

$$F_t\left(\exp\left(2\pi i t \cdot \frac{iz+h}{k}\right)\right) = F_t\left(\exp\left(2\pi i \frac{itz/(t,k) + th/(t,k)}{k/(t,k)}\right)\right) \tag{2.9}$$

$$F_t\left(\exp\left(2\pi i (2t) \cdot \frac{iz+h}{k}\right)\right) = F_t\left(\exp\left(2\pi i \frac{2itz/(2t,k) + 2th/(2t,k)}{k/(2t,k)}\right)\right) \tag{2.10}$$

$$F_t\left(\exp\left(2\pi i (4t) \cdot \frac{iz+h}{k}\right)\right) = F_t\left(\exp\left(2\pi i \frac{4itz/(4t,k) + 4th/(4t,k)}{k/(4t,k)}\right)\right). \tag{2.11}$$

We can then apply (2.5) to the right hand side (after reformulating it in terms of the function f) of each of (2.9), (2.10), (2.11), noting that

$$\left(\frac{th}{(t,k)}, \frac{k}{(t,k)}\right) = \left(\frac{2th}{(2t,k)}, \frac{k}{(2t,k)}\right) = \left(\frac{4th}{(4t,k)}, \frac{k}{(4t,k)}\right) = 1.$$

One then obtains (2.8). In particular, after computation, we have

$$G_{t,h,k}(x) =$$
$$\begin{cases} \dfrac{f^{3t}\left(x^{2^{2d-1}r^2}\exp\left\{2\pi i\frac{2^d r u\left(\frac{th}{2^{d-1}r},\frac{k}{2^d r}\right)-2^{2d-1}r^2h'}{k}\right\}\right)f(x^t)}{f^{2t}\left(x^{2^{2d}r^2}\exp\left\{2\pi i\frac{2^d r u\left(\frac{th}{2^d r},\frac{k}{2^d r}\right)-2^{2d}r^2h'}{k}\right\}\right)f^t\left(x^{2^{2d-2}r^2}\exp\left\{2\pi i\frac{2^d r u\left(\frac{th}{2^d r},\frac{k}{2^d r}\right)-2^{2d-2}r^2h'}{k}\right\}\right)}, & c \geq d \\[2ex] \dfrac{f^{3t}\left(x^{2^{2c-1}r^2}\exp\left\{2\pi i\frac{2^{c+1} r u\left(\frac{th}{2^c r},\frac{k}{2^{c+1} r}\right)-2^{2c+1}r^2h'}{k}\right\}\right)f(x^t)}{f^{2t}\left(x^{2^{2c}r^2}\exp\left\{2\pi i\frac{2^c r u\left(\frac{th}{2^c r},\frac{k}{2^c r}\right)-2^{2c}r^2h'}{k}\right\}\right)f^t\left(x^{2^{2c}r^2}\exp\left\{2\pi i\frac{2^{c+1} r u\left(\frac{th}{2^{c-1} r},\frac{k}{2^{c+1} r}\right)-2^{2c}r^2h'}{k}\right\}\right)}, & c = d-1\ , \\[2ex] \dfrac{f^{3t}\left(x^{2^{2c+1}r^2}\exp\left\{2\pi i\frac{2^{c+1} r u\left(\frac{th}{2^c r},\frac{k}{2^{c+1} r}\right)-2^{2c+1}r^2h'}{k}\right\}\right)f(x^t)}{f^{2t}\left(x^{2^{2c}r^2}\exp\left\{2\pi i\frac{2^c r u\left(\frac{th}{2^c r},\frac{k}{2^c r}\right)-2^{2c}r^2h'}{k}\right\}\right)f^t\left(2^{2^{2c+2}r^2}\exp\left\{2\pi i\frac{2^{c+2} r\left(\frac{th}{2^c r},\frac{k}{2^{c+2} r}\right)-2^{2c+2}r^2h'}{k}\right\}\right)}, & c \leq d-2 \end{cases}$$

where $u(a,b)$ is equal to the unique integer l such that $0 \leq l < b$ and $al \equiv -1 \pmod{b}$. Using knowledge of the fact that the radius of convergence of the power series for f is 1 as well as the lack of any roots of f in the open unit disc, one finds that the radius of convergence of the power series of $G_{t,h,k}$ is at most 1. Furthermore, it follows that the constant term of $G_{t,h,k}$ is 1 because the constant term of f is 1. □

We now present a lemma regarding the power series $G_{t,h,k}$.

Lemma 2.4. *For a given t, the number of possible power series $G_{t,h,k}$ ranging over all h, k with $k \geq 1$, $0 \leq h < k$, and $(h,k) = 1$ is finite.*

Proof. $G_{t,h,k}$ is the image of the modular form F_t under some modular transformation. Since it is a modular transformation under a subgroup of $SL_2(\mathbb{Z})$ with finite index, F_t has finitely many cuspidal Fourier expansions. Therefore, $G_{t,h,k}$ must be one of finitely many power series. □

3. Proof of Theorem 1.1 and Corollaries 1.2, 1.3

3.1. **Review of the Circle Method.** In this paper, we will use the circle method to obtain an exact formula for the coefficients $a_t(n)$ of $F_t(x)$, a function whose poles in the complex plane occur at the roots of unity. For a detailed exposition of this method, see [1]. Another application of the circle method to finding formulas involving partition functions is provided in [2]. Here, we present a brief overview. We begin with the equation

$$\frac{F_t(x)}{x^{n+1}} = \sum_{k=0}^{\infty} \frac{a_t(k)x^k}{x^{n+1}}$$

for $0 < |x| < 1$. Cauchy's residue theorem implies that

$$a_t(n) = \frac{1}{2\pi i}\int_C \frac{F_t(x)}{x^{n+1}}\,dx, \tag{3.1}$$

where C is a simple closed positively oriented contour encircling the origin and lying inside the unit circle. Using the change of variables $x = e^{2\pi i\tau}$, we can replace C by a path in the τ-plane. One possible contour is the horizontal path from i to $i+1$, which corresponds to a counterclockwise circular path of radius $e^{-2\pi}$ in the x-plane.

The key idea is to replace the horizontal path by the Rademacher path $P(N)$ consisting of the arcs of Ford circles obtained from the Farey sequence of order N. Such a path is shown for $N = 4$ in Figure 2.

Thus, (3.1) becomes

$$a_t(n) = \sum_{k=1}^{N}\sum_{\substack{0\leq h<k\\(h,k)=1}}\int_{\gamma(h,k)} F_t(e^{2\pi i\tau})e^{-2\pi i n\tau}\,d\tau, \tag{3.2}$$

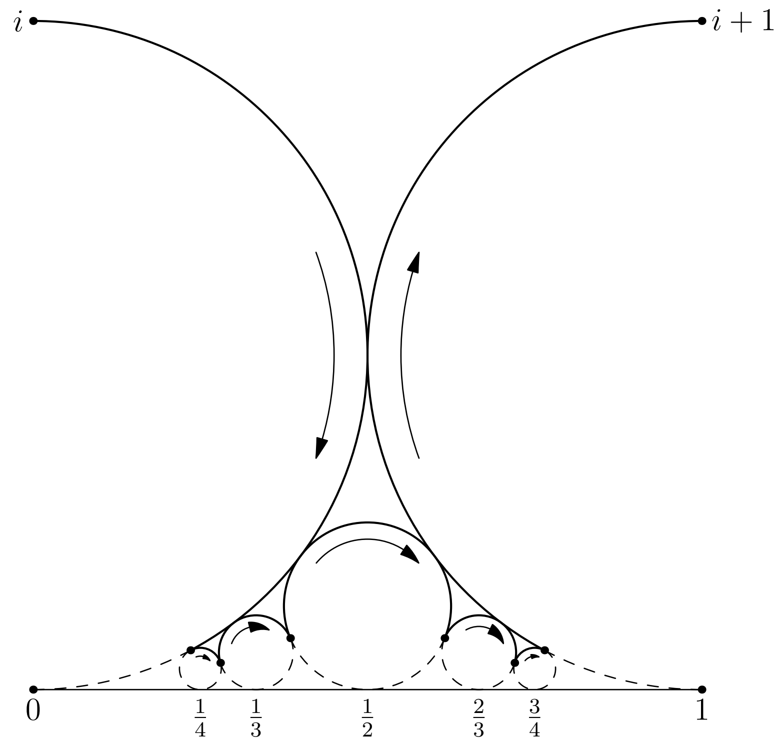

FIGURE 2. The Rademacher path $P(N)$ for $N = 4$

where $\gamma(h,k)$ is the upper arc of the Ford circle $C(h,k)$ that contributes to Rademacher's path of integration. Then, one can make the following additional change of variable:

$$z = -ik^2\left(\tau - \frac{h}{k}\right), \tag{3.3}$$

which maps the Ford circle $C(h,k)$ to a circle K of radius $\frac{1}{2}$ centered at $\frac{1}{2}$ in the z-plane. The arc $\gamma(h,k)$ is transformed into a arc on the new circle going between points $z_1(h,k)$ and $z_2(h,k)$, as shown in Figure 3.

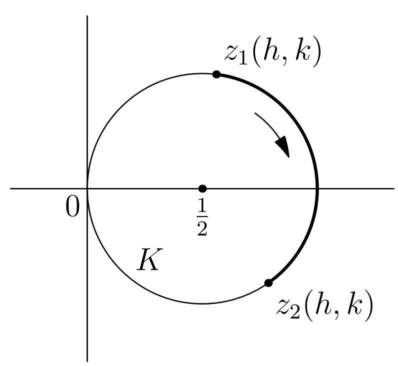

FIGURE 3. Image of the path on a Ford circle under a change of variable

With the substitution (3.3), (3.2) becomes

$$a_t(n) = \sum_{k=1}^{N} \sum_{\substack{0\le h<k \\ (h,k)=1}} ik^{-2}e^{-2\pi inh/k} \int_{z_1(h,k)}^{z_2(h,k)} e^{2n\pi z/k^2} F_t\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right)\right) dz. \tag{3.4}$$

We will now compute the right hand side of (3.4). We use the transformation formula given by Theorem 2.3 with z/k substituted for z:

$$F_t\left(\exp\left(2\pi i\frac{iz/k+h}{k}\right)\right)=\Omega_{t,h,k}\sqrt{\frac{z}{k}}\psi_{t,k}\left(\frac{z}{k}\right)G_{t,h,k}(x')$$

to obtain

$$a_t(n)=\sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-5/2}e^{-2\pi inh/k}\Omega_{t,h,k}\int_{z_1(h,k)}^{z_2(h,k)}\psi_{t,k}\left(\frac{z}{k}\right)G_{t,h,k}(x')e^{2n\pi z/k^2}\sqrt{z}\,dz, \tag{3.5}$$

where $x'=\exp\left(2\pi i\frac{ik/z+h'}{tk}\right)$.

Now, for the remainder of the paper, we will write $c=\mathrm{ord}_2(t)$ and $d=\mathrm{ord}_2(k)$. Let us split the summation over k in (3.5) into three separate sums:

$$a_t(n)=a_t^{(1)}(n)+a_t^{(2)}(n)+a_t^{(3)}(n), \tag{3.6}$$

where

$$a_t^{(1)}(n)=\sum_{\substack{1\le k\le N\\c\ge d}}\sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-5/2}e^{-2\pi inh/k}\Omega_{t,h,k}\int_{z_1(h,k)}^{z_2(h,k)}\psi_{t,k}\left(\frac{z}{k}\right)G_{t,h,k}(x')e^{2n\pi z/k^2}\sqrt{z}\,dz, \tag{3.7}$$

while $a_t^{(2)}$ and $a_t^{(3)}$ are the same as $a_t^{(1)}$ except that the condition $c\ge d$ in the sum on the right hand side of (3.7) is replaced with $c=d-1$ and $c\le d-2$, respectively. Now, we proceed to calculate $a_t^{(1)}$, $a_2^{(2)}$, and $a_3^{(3)}$ individually.

3.2. **Calculation of $a_t^{(1)}(n)$.** First, we make the following observation:

Lemma 3.1. *The quantity $-3\cdot 2^{2d-2}r^2+1$ is positive if and only if $(k,2t)=1$.*

Proof. Note that if $(k,2t)=1$, then $d=0$ and $r=1$, and so, $-3\cdot 2^{2d-2}r^2+1=\frac{1}{4}>0$. Now, suppose that $(k,2t)\ne 1$. Then, either $2|k$, in which case

$$d\ge 1\Rightarrow -3\cdot 2^{2d-2}r^2+1\le -3r^2+1<0,$$

or $2\nmid k$, in which case $(k,t)>1$, and so,

$$r\ge 2\Rightarrow -3\cdot 2^{2d-2}r^2+1\le -3\left(\frac{1}{4}\right)(2)^2+1<0.$$

This completes the proof of the lemma. □

Now, write the Taylor expansion

$$G_{t,h,k}(x)=\sum_{j=0}^{\infty}b_{t,h,k}(j)x^j. \tag{3.8}$$

We split into two cases, depending on the value of $(k,2t)$. First, we consider the case in which $(k,2t)>1$. Note that we can move the circular path of the integral in (3.7) to the path along the chord indicated in Figure 4 without affecting the value. Thus, we shall do so, and let us estimate the integrand of (3.7) along the chord. Using Lemma 3.1 and the fact that $\Re\left(\frac{1}{z}\right)\ge 1$ for z on the chord, we have

$$\left|\psi_{t,k}\left(\frac{z}{k}\right)G_{t,h,k}\left(\exp\left(2\pi i\frac{ik/z+h'}{tk}\right)\right)e^{2n\pi z/k^2}\sqrt{z}\right|$$

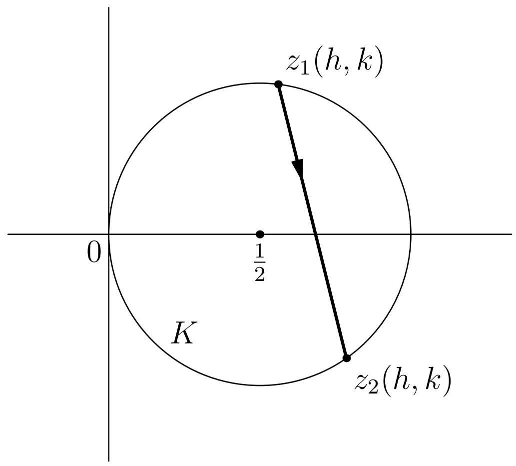

FIGURE 4. Path of integration along the chord connecting $z_1(h,k)$ and $z_2(h,k)$

$$\begin{aligned}
&= |z|^{1/2}2^{t/2}\exp\left\{\frac{\pi(-3\cdot 2^{2d-2}r^2+1)}{12}\Re\left(\frac{1}{z}\right)-\frac{\pi}{12k^2}\Re(z)\right\} \\
&\quad \times e^{2n\pi\Re(z)/k^2}\left|\sum_{m=0}^{\infty} b_{t,h,k}(m)e^{2\pi ih'm/tk}e^{-2\pi m/tz}\right| \\
&\leq |z|^{1/2}2^{t/2}\exp\left\{\frac{\pi(-3\cdot 2^{2d-2}r^2+1)}{12}\Re\left(\frac{1}{z}\right)\right\}e^{2n\pi/k^2}\sum_{m=0}^{\infty}|b_{t,h,k}(m)|e^{-\frac{2\pi m}{t}\Re\left(\frac{1}{z}\right)} \\
&\leq |z|^{1/2}2^{t/2}e^{2n\pi}\sum_{m=0}^{\infty}|b_{t,h,k}(m)|e^{-\frac{2\pi}{t}\left(m+\frac{t}{24}(3\cdot 2^{2d-2}r^2-1)\right)} \\
&\leq |z|^{1/2}2^{t/2}\exp\left\{\frac{\pi}{12}\right\}e^{2n\pi}\sum_{m=0}^{\infty}|b_{t,h,k}(m)|e^{-\frac{2\pi}{t}m} \\
&= b|z|^{1/2},
\end{aligned}$$

for some nonnegative constant b not depending on h, k, z, or N (since the summation in the penultimate line converges due to Theorem 2.3 and is bounded by a constant not depending on h or k due to Lemma 2.4). Thus, as arguing in [1], we have that

$$\left|\int_{z_1(h,k)}^{z_2(h,k)}\psi_{t,k}\left(\frac{z}{k}\right)G_{t,h,k}(x')e^{2n\pi z/k^2}\sqrt{z}\,dz\right| = O(k^{3/2}N^{-3/2})$$

in the case $(k,2t)>1$, and so,

$$\left|\sum_{\substack{1\leq k\leq N\\(k,2t)>1}}\sum_{\substack{0\leq h<k\\(h,k)=1}} ik^{-5/2}e^{-2\pi inh/k}\Omega_{t,h,k}\int_{z_1(h,k)}^{z_2(h,k)}\psi_{t,k}\left(\frac{z}{k}\right)G_t(x')e^{2n\pi z/k^2}\sqrt{z}\,dz\right| \tag{3.9}$$

$$\begin{aligned}
&\leq \sum_{\substack{1\leq k\leq N\\c\geq d}}\sum_{\substack{0\leq h<k\\(h,k)=1}} O(k^{-1}N^{-3/2}) \\
&\leq \sum_{\substack{1\leq k\leq N\\c\geq d}} O(N^{-3/2}) = O(N^{-1/2}).
\end{aligned}$$

Next, consider the case in which $(k, 2t) = 1$. In this case, $d = 0$ and $r = 1$, and thus, $-3 \cdot 2^{2d-2}r^2 + 1 = \frac{1}{4}$, as shown in the proof of Lemma 3.1. Define

$$M(t,k) = \left\lceil \frac{t}{96} \right\rceil - 1 \tag{3.10}$$

whenever t and k satisfy $c \geq d$. Let us split up $G_{t,h,k}$ as

$$G_{t,h,k}(x) = G^{(1)}_{t,h,k}(x) + G^{(2)}_{t,h,k}(x), \tag{3.11}$$

where

$$G^{(1)}_{t,h,k}(x) = \sum_{j=0}^{M(t,k)} b_{t,h,k}(j)x^j \quad \text{and} \quad G^{(2)}_{t,h,k}(x) = \sum_{j=M(t,k)+1}^{\infty} b_{t,h,k}(j)x^j. \tag{3.12}$$

Also let

$$\Psi = \psi_{t,k}\left(\frac{z}{k}\right) e^{2n\pi z/k^2}\sqrt{z}. \tag{3.13}$$

Now, let us estimate $\left|\Psi G^{(2)}_{t,h,k}(x')\right|$ along the chord in Figure 4:

$$\begin{aligned}
\left|\Psi G^{(2)}_{t,h,k}(x')\right| &= |z|^{1/2}2^{t/2}\exp\left\{\frac{\pi}{48}\Re\left(\frac{1}{z}\right) - \frac{\pi}{12k^2}\Re(z)\right\} \\
&\quad \times e^{2n\pi\Re(z)/k^2}\left|\sum_{m=M(t,k)+1}^{\infty} b_{t,h,k}(m)e^{2\pi i h' m/tk}e^{-2\pi m/tz}\right| \\
&\leq |z|^{1/2}2^{t/2}e^{2n\pi}\sum_{m=M(t,k)+1}^{\infty} |b_{t,h,k}(m)|e^{-\frac{2\pi}{t}\left(m-\frac{t}{96}\right)\Re\left(\frac{1}{z}\right)} \\
&\leq |z|^{1/2}2^{t/2}e^{2n\pi}\sum_{m=M(t,k)+1}^{\infty} |b_{t,h,k}(m)|e^{-\frac{2\pi}{t}\left(m-\frac{t}{96}\right)} \\
&= |z|^{1/2}2^{t/2}\exp\left\{\frac{\pi}{48}\right\}e^{2n\pi}\sum_{m=M(t,k)+1}^{\infty} |b_{t,h,k}(m)|e^{-\frac{2\pi}{t}m},
\end{aligned}$$

using the fact that $m - \frac{t}{96} > 0$ for integers $m > M(t,k)$ (by our choice of $M(t,k)$). Then, as before, we have $\left|\Psi G^{(2)}_{t,h,k}\right| \leq b|z|^{1/2}$ for a constant b not depending on z or N, and it follows that

$$\sum_{\substack{1\leq k\leq n\\(k,2t)=1}}\sum_{\substack{0\leq h<k\\(h,k)=1}} ik^{-5/2}e^{-2\pi nh/k}\Omega_{t,h,k}\int_{z_1(h,k)}^{z_2(h,k)} \Psi G^{(2)}_{t,h,k}(x')\,dz = O(N^{-1/2}). \tag{3.14}$$

Now,

$$\begin{aligned}
I &= \int_{z_1(h,k)}^{z_2(h,k)} \Psi G^{(1)}_{t,h,k}(x')\,dz \\
&= \int_{K(-)} \Psi G^{(1)}_{t,h,k}(x')\,dz - \int_0^{z_1(h,k)} \Psi G^{(1)}_{t,h,k}(x')\,dz - \int_{z_2(h,k)}^{0} \Psi G^{(1)}_{t,h,k}(x')\,dz, \qquad (3.15)
\end{aligned}$$

where $K(-)$ indicates that the integration occurs once around K in the clockwise direction, while the other two integrals occur clockwise around K between the appropriate points. Recall that the length of the arc going from 0 to $z_1(h,k)$ is bounded by $|z_1(h,k)| < \sqrt{2}\frac{k}{N}$ (see [1]). Then, note that for z on K, we have $0 \leq \Re(z) \leq 1$ and $\Re(1/z) = 1$, meaning that

$$\begin{aligned}
|\Psi| &= \left|\psi_{t,k}\left(\frac{z}{k}\right)e^{2n\pi z/k^2}\sqrt{z}\right| \\
&= 2^{t/2}e^{2n\pi\Re(z)/k^2}|z|^{1/2}\exp\left\{\frac{\pi}{12}\left(\frac{1}{4}\Re\left(\frac{1}{z}\right) - \frac{\Re(z)}{k^2}\right)\right\} \\
&\leq \frac{e^{2n\pi}2^{(1+2t)/4}k^{1/2}e^{\pi/48}}{N^{1/2}}. \qquad (3.16)
\end{aligned}$$

Also for z on the chord,

$$\begin{aligned}\left|G^{(1)}_{t,h,k}(x')\right| &\leq \sum_{m=0}^{M(t,k)} |b_{t,h,k}(m)||x'|^m \\ &= \sum_{m=0}^{M(t,k)} |b_{t,h,k}(m)| \left|-\frac{2\pi}{t}\Re\left(\frac{1}{z}\right)\right|^m \\ &= \sum_{m=0}^{M(t,k)} |b_{t,h,k}(m)| \left(\frac{2\pi}{t}\right)^m. \qquad (3.17)\end{aligned}$$

Combining (3.16) and (3.17) shows that $\left|\Psi G^{(1)}_{t,h,k}(x')\right| = O(N^{-1/2})$. Using the fact that $|z| < \sqrt{2}\frac{k}{N}$ on the arc from 0 to $z_1(h,k)$, we have

$$\int_0^{z_1(h,k)} \Psi G^{(1)}_{t,h,k}(x')\, dz = O(N^{-3/2}).$$

As before, this gives rise to

$$\sum_{\substack{1\leq k\leq N\\(k,2t)=1}} \sum_{\substack{0\leq h<k\\(h,k)=1}} ik^{-5/2}e^{-2\pi inh/k}\Omega_{t,h,k}\int_0^{z_1(h,k)} \Psi G^{(1)}_{t,h,k}(x')\, dz = O(N^{-1/2}). \tag{3.18}$$

Similarly,

$$\sum_{\substack{1\leq k\leq N\\(k,2t)=1}} \sum_{\substack{0\leq h<k\\(h,k)=1}} ik^{-5/2}e^{-2\pi inh/k}\Omega_{t,h,k}\int_{z_2(h,k)}^{0} \Psi G^{(1)}_{t,h,k}(x')\, dz = O(N^{-1/2}). \tag{3.19}$$

Now,

$$\int_{K(-)} \Psi G^{(1)}_{t,h,k}(x')\, dz = 2^{t/2}\sum_{m=0}^{M(t,k)} b_{t,h,k}(m)\int_{K(-)} z^{1/2}\exp\left\{\frac{\pi\left(\frac{1}{4}-\frac{24m}{t}\right)}{12z} + \frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right) + 2\pi i\frac{mh'}{tk}\right\} dz.$$

We now change variables and put $u = \frac{w}{z}$, where $w = \frac{\pi\left(\frac{1}{4}-\frac{24m}{t}\right)}{12}$. The above expression then becomes

$$-2^{t/2}\sum_{m=0}^{M(t,k)} w^{3/2}b_{t,h,k}e^{2\pi imh'/tk}\int_{w-\infty i}^{w+\infty i} u^{-5/2}\exp\left\{u+\frac{2\pi w}{k^2}\left(n-\frac{1}{24}\right)\frac{1}{u}\right\}. \tag{3.20}$$

Note that $w > 0$ for $0 \leq m \leq M(t,k)$. Now, we make use of the following formula for a modified Bessel function of the first kind found on page 181 of [12]:

$$I_v(y) = \frac{\left(\frac{y}{2}\right)^v}{2\pi i}\int_{w-\infty i}^{w+\infty i} u^{-v-1}e^{u+(y^2/4u)}\, du \quad (\text{for } w>0,\ \Re(v)>0) \tag{3.21}$$

If we take $v = 3/2$ and

$$y = 2\sqrt{\frac{2\pi w}{k^2}\left(n-\frac{1}{24}\right)}, \tag{3.22}$$

then substituting (3.21) into (3.20) transforms (3.20) into

$$-(2\pi i)2^{t/2}\sum_{m=0}^{M(t,k)} w^{3/2}b_{t,h,k}(m)e^{2\pi imh'/tk}\left(\frac{y}{2}\right)^{-3/2} I_{3/2}(y) = \sum_{m=0}^{M(t,k)} -\frac{2^{t/2}(2\pi i)\left(\frac{1}{4}-\frac{24m}{t}\right)^{3/4}\left(n-\frac{1}{24}\right)^{-3/4}}{6^{-3/4}12^{3/2}k^{-3/2}} b_{t,h,k}(m)e^{2\pi imh'/tk}I_{3/2}(y).$$

Combining this with (3.9), (3.14), (3.15), (3.18), and (3.19) yields

$$a_t^{(1)}(n) = O(N^{-1/2}) + \sum_{\substack{1\le k\le N\\(k,2t)=1}} \sum_{\substack{0\le h<k\\(h,k)=1}} \sum_{m=0}^{M(t,k)} k^{-1} e^{2\pi i(mh'/tk - nh/k)} \Omega_{t,h,k} b_{t,h,k} \frac{2^{t/2}(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} \times \left(\frac{1}{4}-\frac{24m}{t}\right)^{3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(\frac{1}{4}-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)}\right). \tag{3.23}$$

3.3. **Calculation of $a_t^{(2)}(n)$.** As in the case of calculating $a_t^{(1)}(n)$, we write (3.8) and use the definitions (3.11), (3.12), and (3.13). Now, define

$$M(t,k) = \left\lceil \frac{(3\cdot 2^{2c}r^2+1)t}{24} \right\rceil - 1 \tag{3.24}$$

whenever t and k are numbers for which $c = d-1$. Then,

$$a_t^{(2)}(n) = \sum_{\substack{1\le k\le N\\c=d-1}} \sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \left(\int_{z_1(h,k)}^{z_2(h,k)} \Psi G_{t,h,k}^{(1)}(x')\,dz + \int_{z_1(h,k)}^{z_2(h,k)} \Psi G_{t,h,k}^{(2)}(x')\,dz \right). \tag{3.25}$$

As in the case for $a_t^{(1)}(n)$,

$$\sum_{\substack{1\le k\le N\\c=d-1}} \sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \int_{z_1(h,k)}^{z_2(h,k)} \Psi G_{t,h,k}^{(2)}(x')\,dz = O(N^{-1/2}). \tag{3.26}$$

using a similar line of reasoning.

As before, we may write (3.15). Using the same method used to derive (3.18) and (3.19), we obtain

$$\sum_{\substack{1\le k\le N\\c=d-1}} \sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \int_{0}^{z_1(h,k)} \Psi G_{t,h,k}^{(1)}(x')\,dz = O(N^{-1/2}) \tag{3.27}$$

$$\sum_{\substack{1\le k\le N\\c=d-1}} \sum_{\substack{0\le h<k\\(h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \int_{z_2(h,k)}^{0} \Psi G_{t,h,k}^{(1)}(x')\,dz = O(N^{-1/2}). \tag{3.28}$$

Next, note that if $c = d-1$, then

$$\int_{K(-)} \Psi G_{t,h,k}^{(1)}(x')\,dz = 2^{-t/2} \sum_{m=0}^{M(t,k)} b_{t,h,k}(m) \int_{K(-)} z^{1/2} \exp\left\{ \frac{\pi\left(3\cdot 2^{2c}r^2+1-\frac{24m}{t}\right)}{12z} + \frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right) + 2\pi i \frac{mh'}{tk} \right\} dz. \tag{3.29}$$

We now change variables and put $u = \frac{w}{z}$, where $w = \frac{\pi(3\cdot 2^{2c}r^2+1-\frac{24m}{t})}{12}$. Taking $v = \frac{3}{2}$ and (3.22) in (3.21), we see that (3.29) is equal to

$$-(2\pi i)2^{-t/2} \sum_{m=0}^{M(t,k)} w^{3/2} b_{t,h,k}(m) e^{2\pi imh'/tk} \left(\frac{y}{2}\right)^{-3/2} I_{3/2}(y)$$
$$= \sum_{m=0}^{M(t,k)} -\frac{2^{-t/2}(2\pi i)\left(3\cdot 2^{2c}r^2+1-\frac{24m}{t}\right)^{-3/4}\left(n-\frac{1}{24}\right)^{-3/4}}{6^{-3/4}12^{3/2}k^{-3/2}} b_{t,h,k}(m) e^{2\pi imh'/tk} I_{3/2}(y),$$

and so, combining this with (3.25), (3.26), (3.15), (3.27), and (3.28) yields

$$a_t^{(2)}(n) = O(N^{-1/2}) + \sum_{\substack{1\le k\le N\\ c=d-1}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} k^{-1} e^{2\pi i(mh'/tk - nh/k)} \Omega_{t,h,k} b_{t,h,k}(m) \frac{2^{-t/2}(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} \times \left(3\cdot 2^{2c}r^2 + 1 - \frac{24m}{t}\right)^{3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(3\cdot 2^{2c}r^2+1-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)}\right). \tag{3.30}$$

3.4. **Calculation of $a_t^{(3)}(n)$.** Again, we write (3.8) and use the definitions (3.11), (3.12), and (3.13). However, for t,k satisfying $c\le d-2$, we define

$$M(t,k) = \left\lceil \frac{t}{24} \right\rceil - 1. \tag{3.31}$$

Then,

$$a_t^{(3)}(n) = \sum_{\substack{1\le k\le N\\ c\le d-2}} \sum_{\substack{0\le h<k\\ (h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \left(\int_{z_1(h,k)}^{z_2(h,k)} \Psi G_{t,h,k}^{(1)}(x')\,dz + \int_{z_1(h,k)}^{z_2(h,k)} \Psi G_{t,h,k}^{(2)}(x')\,dz \right). \tag{3.32}$$

As in the case for $a_t^{(1)}(n)$ and $a_t^{(2)}(n)$,

$$\sum_{\substack{1\le k\le N\\ c\le d-2}} \sum_{\substack{0\le h<k\\ (h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \int_{z_1(h,k)}^{z_2(h,k)} \Psi G_{t,h,k}^{(2)}(x')\,dz = O(N^{-1/2}), \tag{3.33}$$

using a similar argument.

Now, as before, we may write (3.15) and obtain

$$\sum_{\substack{1\le k\le N\\ c\le d-2}} \sum_{\substack{0\le h<k\\ (h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \int_{0}^{z_1(h,k)} \Psi G_{t,h,k}^{(1)}(x')\,dz = O(N^{-1/2}) \tag{3.34}$$

$$\sum_{\substack{1\le k\le N\\ c\le d-2}} \sum_{\substack{0\le h<k\\ (h,k)=1}} ik^{-5/2} e^{-2\pi inh/k} \Omega_{t,h,k} \int_{z_2(h,k)}^{0} \Psi G_{t,h,k}^{(1)}(x')\,dz = O(N^{-1/2}). \tag{3.35}$$

Next, note that if $c\le d-2$, then

$$\int_{K(-)} \Psi G_{t,h,k}^{(1)}(x')\,dz = \sum_{m=0}^{M(t,k)} b_{t,h,k}(m) \int_{K(-)} z^{1/2} \exp\left\{ \frac{\pi\left(1-\frac{24m}{t}\right)}{12z} + \frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right) + 2\pi i\frac{mh'}{tk} \right\} dz \tag{3.36}$$

Let us change variables and put $u=\frac{w}{z}$, with $w=\frac{\pi\left(1-\frac{24m}{t}\right)}{12}$. Taking $v=\frac{3}{2}$ and (3.22) in (3.21), we observe that (3.36) is equal to

$$-(2\pi i)\sum_{m=0}^{M(t,k)} w^{3/2} b_{t,h,k}(m) e^{2\pi imh'/tk} \left(\frac{y}{2}\right)^{-3/2} I_{3/2}(y) = \sum_{m=0}^{M(t,k)} -\frac{(2\pi i)\left(1-\frac{24m}{t}\right)^{3/4}\left(n-\frac{1}{24}\right)^{-3/4}}{6^{-3/4}12^{3/2}k^{-3/2}} b_{t,h,k}(m) e^{2\pi imh'/tk} I_{3/2}(y).$$

Therefore, combining the above expression with (3.32), (3.33), (3.15), (3.34), and (3.35) yields

$$a_t^{(3)}(n) = O(N^{-1/2}) + \sum_{\substack{1\le k\le N\\ c\le d-2}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} k^{-1} e^{2\pi i(mh'/tk - nh/k)} \Omega_{t,h,k} b_{t,h,k}(m) \frac{(2\pi)\left(n-\frac{1}{24}\right)^{-3/4}}{24^{3/4}} \times \left(1-\frac{24m}{t}\right)^{3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(1-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)}\right). \tag{3.37}$$

Combining (3.6), (3.23), (3.30), and (3.37) while letting $N \to \infty$ gives us Theorem 1.1.

3.5. **Proof of Corollaries.** Now, we present proofs of the corollaries. First, let us introduce a lemma:

Lemma 3.2. *If $0 < \alpha < \beta$, then*

$$\lim_{n\to\infty} \frac{I_{3/2}(\alpha n)}{I_{3/2}(\beta n)} = 0.$$

Proof. Using (1.3), we have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{I_{3/2}(\alpha n)}{I_{3/2}(\beta n)} &= \lim_{n\to\infty} \frac{\frac{e^{\alpha n}}{\sqrt{2\pi\alpha n}}}{\frac{e^{\beta n}}{\sqrt{2\pi\beta n}}}\\
&= \lim_{n\to\infty} \sqrt{\frac{\beta}{\alpha}} e^{(\alpha-\beta)n}\\
&= 0.
\end{aligned}$$

□

We will also note the following lemma:

Lemma 3.3. *The integral*

$$\int_1^\infty I_{3/2}\left(\frac{1}{x}\right) dx$$

converges.

We shall omit the proof of the lemma here, but it can be proven by using (1.9). Now, we are ready to prove the desired corollaries.

Proof of Corollary 1.2: First, note that if $c \ge d$ and $m \ge 0$, then

$$\begin{aligned}
\frac{\pi}{k}\sqrt{\frac{2}{3}\left(\frac{1}{4}-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)} &\le \frac{\pi}{k}\sqrt{\frac{1}{6}\left(n-\frac{1}{24}\right)}\\
&< \frac{\pi}{k}\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}.
\end{aligned} \tag{3.38}$$

Also, if $c = d-1$ and $m \ge 0$, then

$$\begin{aligned}
\frac{\pi}{k}\sqrt{\frac{2}{3}\left(3\cdot 2^{2c}r^2+1-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)} &\le \frac{\pi}{k}\sqrt{\frac{2}{3}\left(3\cdot 2^{2c}r^2+1\right)\left(n-\frac{1}{24}\right)}\\
&= \pi\sqrt{\left(\frac{1}{2}\left(\frac{2^{c+1}r}{k}\right)^2+\frac{2}{3k^2}\right)\left(n-\frac{1}{24}\right)}\\
&\le \pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}.
\end{aligned} \tag{3.39}$$

Observe that equality occurs if and only if $m = 0$, $k = 2^{c+1}$, and $r = 1$. Similarly, observe that if $c \geq d-2$ and $m \geq 0$, then

$$\begin{aligned}\frac{\pi}{k}\sqrt{\frac{2}{3}\left(1-\frac{24m}{t}\right)\left(n-\frac{1}{24}\right)} &\leq \pi\sqrt{\frac{2}{3k^2}\left(n-\frac{1}{24}\right)} \\ &\leq \pi\sqrt{\frac{2}{3\cdot(2^{c+2})^2}\left(n-\frac{1}{24}\right)} \\ &< \pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}. \end{aligned} \tag{3.40}$$

Next, note that for $m \geq 0$, we have

$$\begin{aligned} 3\cdot 2^{2c}r^2+1-\frac{24m}{t} &\leq 3\cdot 2^{2c}r^2+1 \\ &= 3(2^c r)^2+1 \\ &\leq 3t^2+1 = C_1. \end{aligned} \tag{3.41}$$

Moreover, by (3.24) and (3.31), we see that for any postive integer k satisfying $c \leq d-1$,

$$\begin{aligned} M(t,k) &\leq \frac{t}{24}\left(3\cdot 2^{2c}r^2+1\right) \\ &\leq \frac{t}{24}(3t^2+1) = \frac{t}{24}C_1, \end{aligned}$$

using (3.41). Also, by (3.10), it is easy to verify that

$$M(t,k) \leq \frac{t}{24}C_1 \tag{3.42}$$

holds when $c \geq d$ also. Let C be a constant such that $|b_{t,h,k}(m)| \leq C$ for all h, k, and m with $(h,k) = 1$ and $0 \leq m \leq \frac{(3t^2+1)t}{24}$ (such a constant C is guaranteed to exist by Lemma 2.4). Let $K = \frac{2\pi}{24^{3/4}}$.

For k such that $c = d-1$, we have, using (3.41) and (3.42),

$$\begin{aligned} &\sum_{\substack{0\leq h<k\\(h,k)=1}}\sum_{m=0}^{M(t,k)}\left|g_t^{(2)}(h,k,m,n)\right| \\ &\leq \sum_{\substack{0\leq h<k\\(h,k)=1}}\sum_{m=0}^{M(t,k)} 2^{-t/2}k^{-1}CKC_1^{3/4}\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right) \\ &\leq \sum_{\substack{0\leq h<k\\(h,k)=1}}\left(\frac{t}{24}C_1+1\right)\frac{CK}{2^{t/2}k}C_1^{3/4}\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right) \\ &\leq \left(\frac{t}{24}C_1+1\right)\frac{CK}{2^{t/2}}C_1^{3/4}\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right) \\ &= C_2\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right), \end{aligned} \tag{3.43}$$

where C_2 is a constant dependent only on t.

Next, choose a positive integer constant D such that

$$\frac{\pi}{k}\sqrt{\frac{2}{3}C_1} < \pi\sqrt{\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}} \tag{3.44}$$

for all integers $k > D$. Then, using (3.43), we have

$$\sum_{\substack{D+1\le k\\ c=d-1}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} \frac{\left|g_t^{(2)}(h,k,m,n)\right|}{\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)} \tag{3.45}$$

$$\le C_2 \sum_{k=D+1}^{\infty} \frac{I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)}.$$

Letting

$$f_n(x) = \frac{I_{3/2}\left(\frac{\pi}{x}\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)},$$

we note that the sum in (3.45) is bounded from above by

$$C_2\int_D^\infty f_n(x)\,dx = \frac{C_2\pi\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}}{I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)} \int_{D\left(\pi\sqrt{\frac{2}{3}C_1\left(n-\frac{1}{24}\right)}\right)^{-1}}^\infty I_{3/2}\left(\frac{1}{u}\right)\,du,$$

which converges due to Lemma 3.3. Now, observe that $\lim_{n\to\infty} f_n(x) = 0$, by (3.44) and Lemma 3.2. Furthermore, the convergence is uniform in $[1,+\infty)$. Thus,

$$\lim_{n\to\infty}\int_1^\infty f_n(x)\,dx = \int_1^\infty \lim_{n\to\infty} f_n(x)\,dx = \int_1^\infty 0\,dx = 0,$$

and it follows that

$$\lim_{n\to\infty} \sum_{\substack{D+1\le k\\ c=d-1}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} \frac{|g_t^{(2)}(h,k,m,n)|}{\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)} = 0. \tag{3.46}$$

Moreover,

$$\lim_{n\to\infty} \sum_{\substack{1\le k\le D\\ c=d-1}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} \frac{|g_t^{(2)}(h,k,m,n)|}{\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)} \tag{3.47}$$

$$= \frac{\pi(3\cdot 2^{2c}+1)^{3/4} B_{2^{c+1}}(n)}{2^{c+\frac{t}{2}+\frac{9}{4}}\cdot 3^{3/4}},$$

using (3.39), (3.41), Lemma 3.2, $b_{t,h,2^{c+1}}(0) = 1$, and the fact that the above sum is finite. In a similar fashion to (3.46), one can use (3.38) and (3.39) to obtain

$$\lim_{n\to\infty} \sum_{\substack{1\le k\\ (k,2t)=1}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} \frac{\left|g_t^{(1)}(h,k,m,n)\right|}{\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)} = 0) \tag{3.48}$$

$$\lim_{n\to\infty} \sum_{\substack{1\le k\\ c\le d-2}} \sum_{\substack{0\le h<k\\ (h,k)=1}} \sum_{m=0}^{M(t,k)} \frac{\left|g_t^{(3)}(h,k,m,n)\right|}{\left(n-\frac{1}{24}\right)^{-3/4} I_{3/2}\left(\pi\sqrt{\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)}\right)} = 0. \tag{3.49}$$

Corollary 1.2 now follows from (3.46), (3.47), (3.48), (3.49), and (1.3).

Proof of Corollary 1.3: Recall that if t is good, then $B_{2^{c+1}}(n) \neq 0$ for all n. Moreover, examining (1.10) shows that $B_{2^{c+1}}(n+2^{c+1}) = B_{2^{c+1}}(n)$ for all n. Therefore,

$$\min_{n\ge 0}\left|\frac{3^{-1/2}(3\cdot 2^{2c}+1)^{1/2} B_{2^{c+1}}(n)}{2^{\frac{c}{2}+\frac{t}{2}+\frac{5}{2}}}\right| > 0.$$

Corollary 1.2 then implies that

$$a_t(n) \sim \frac{3^{-1/2}(3\cdot 2^{2c}+1)^{1/2}B_{2^{c+1}}(n)}{2^{\frac{c}{2}+\frac{t}{2}+\frac{5}{2}}\left(n-\frac{1}{24}\right)}e^{\pi\sqrt{\left(\left(\frac{1}{2}+\frac{1}{3\cdot 2^{2c+1}}\right)\left(n-\frac{1}{24}\right)\right)}}.$$

Thus, for sufficiently large n, the sign of $a_t^{even}(n) - a_t^{odd}(n)$ is given precisely by the sign of $B_{2^{c+1}}(n)$. However, it is easily verified from the definition (1.10) that $B_{2^{c+1}}(n+2^c) = -B_{2^{c+1}}(n)$. The desired conclusion now follows. □

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References

[1] T. M. Apostol. *Modular functions and Dirichlet series in number theory.* Springer-Verlag, 1997.
[2] K. Bringmann. Asymptotic formulas for some restricted partition functions. *Ramanujan J.*, 12(2):257–266, 2006.
[3] F. Garvan, D. Kim, and D. Stanton. Cranks and t-cores. *Inventiones Mathematicae*, 101:1–17, 1990.
[4] G.-N. Han. The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications. *arXiv:0805.1398*, 2008. preprint.
[5] G. H. Hardy and S. Ramanujan. Asymptotic formulae in combinatory analysis. *Proc. London Math. Soc.*, 17:75–115, 1918.
[6] H. Iwaniec. *Spectral methods of automorphic forms.* The American Mathematical Society, 2002.
[7] G. James and A. Kerber. *The representation theory of the symmetric group.* Addison-Wesley, 1961.
[8] N. A. Nekrasov and A. Okounkov. Seiberg-Witten theory and random partitions, 2006.
[9] H. Rademacher. A convergent series for the partition function $p(n)$. *Proc. Nat. Acad. Sci.*, (23):78–84, 1937.
[10] H. Rademacher. On the partition function $p(n)$. *Proc. Nat. Acad. Sci.*, 2(43):241–254, 1937.
[11] B. E. Sagan. *The symmetric group: representations, combinatorial algorithms, and symmetric functions.* Springer-Verlag, 2001.
[12] G. N. Watson. *A treatise on the theory of Bessel functions.* The University Press, 1922.

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